AMSI VACATION RESEARCH SCHOLARSHIPS 2017-2018

Estimating constants in generalised Wente-type estimates

Asama Qureshi

Supervised by Dr. Yann Bernard, Dr. Ting-Ying Chang

Monash University

Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

Australian Government

Department of Education and Training

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

1 Introduction

Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Given two functions a and b such that

$$\nabla a \in L^2(\Omega) \text{ and } \nabla b \in L^2(\Omega),$$

then let ϕ be the unique solution in $L^2(\Omega)$ to the Dirichlet problem

$$\begin{cases} -\Delta\phi &= a_x b_y - a_y b_x \quad \text{in } \Omega \\ \quad \phi &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is parameterised by x and y, and subscript x and y denote the partial derivatives of a and b. The Wente (1969) inequality states that there exists a constant $C_0(\Omega)$ such that

$$\|\phi\|_{L^\infty(\Omega)} + \|\nabla\phi\|_{L^2(\Omega)} \le C_0(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}\,. \tag{2}$$

In this paper, we examine the following generalisation of the Wente inequality on D^2, where D^2 a disk of radius 1 in $\mathbb{R}^2$ centered at the origin,

$$\|\phi\|_{L^\infty(D^2)} + \|\nabla\phi\|_{L^2(D^2)} \le C_0(p, D^2)\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)}\,, \tag{3}$$

with $\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p < \infty$. We find the best constant $C_0(p, D^2)$ in the following result.

Theorem - ϕ satisfies the generalised Wente inequalities:

$$\begin{aligned} \|\phi\|_{L^\infty(D^2)} &\le C_\infty(p, D^2)\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)} \quad \text{and} \\ \|\nabla\phi\|_{L^2(D^2)} &\le C_2(p, D^2)\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)}\,, \end{aligned} \tag{4}$$

for two optimal constants $C_\infty(p, D^2)$ and $C_2(p, D^2)$ satisfying:

$$C_\infty(p, D^2) = \frac{K_p}{2\pi}, \quad C_2(p, D^2) = \sqrt{\frac{K_p}{2\pi}},$$

where

$$K_p = \frac{p\sin(\pi/p)}{(p-1)^{1/p}}, \quad 1 < p < \infty.$$

□

This is a significant result because (1) initially only tells us that $\Delta\phi$ is in L^1, but this does not imply that $\phi \in L^\infty$ or that $\nabla\phi \in L^2$. The "div-curl" structure of (1) with the presence of the negative sign introduces a compensation phenomenon, allowing us to make the estimates in (2).
Baraket (1996) and Hélein (2002) found the optimal constant $C_0(\Omega) = \frac{1}{2\pi} + \sqrt{\frac{1}{2\pi}}$, independently of the domain Ω. That is, Baraket found this constant for simply connected Ω, and Hélein found this for any Ω.

2 Proof of results

Here we will prove the results stated in (4). First we find $C_\infty(p, D^2)$ and then we will find $C_2(p, D^2)$.

Theorem - Let ϕ be the solution to the Dirichlet problem

$$\begin{cases} -\Delta\phi &= a_x b_y - a_y b_x \quad \text{in } D^2 \\ \phi &= 0 \quad \text{on } \partial D^2. \end{cases} \tag{5}$$

Then it holds:

$$\|\phi\|_{L^\infty(D^2)} \leq C_\infty(p, D^2)\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)},$$

with

$$C_\infty(p, D^2) = \frac{K_p}{2\pi}, \quad \text{where } K_p = \frac{p\sin(\pi/p)}{(p-1)^{1/p}}, \quad 1 < p < \infty.$$

Proof

By Green's Representation Theorem we have the following expression for ϕ,

$$\phi(\mathbf{x}_0) = \int_{\partial D^2}\left(\phi\frac{\partial E(\mathbf{x}-\mathbf{x}_0)}{\partial v} - E(\mathbf{x}-\mathbf{x}_0)\frac{\partial\phi}{\partial v}\right)ds + \int_{D^2} E(\mathbf{x}-\mathbf{x}_0)\Delta\phi dx, \tag{6}$$

for fixed $\mathbf{x}_0 \in D^2$, $E(\mathbf{x}-\mathbf{x}_0) = \frac{1}{2\pi}log|\mathbf{x}-\mathbf{x}_0|$ and v is the exterior normal vector to ∂D^2. Choosing $\mathbf{x}_0 = 0$ in (6) gives us

$$\phi(0) = \int_{B_1} E(\mathbf{x})\Delta\phi dx. \tag{7}$$

Lemma - Performing a change of variables on (5) into polar coordinates gives

$$-\Delta\phi = \frac{1}{r}(a_r b_\theta - a_\theta b_r), \tag{8}$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \text{Arctan}(\frac{y}{x})$.

Proof

We start out with (5) and use the chain rule,

$$\begin{aligned} a_x b_y - a_y b_x &= (a_r r_x + a_\theta\theta_x)(b_r r_y + b_\theta\theta_y) - (a_r r_y + a_\theta\theta_y)(b_r r_x + b_\theta\theta_x) \\ &= a_r b_\theta r_x\theta_y + a_\theta b_r r_y\theta_x - a_r b_\theta r_y\theta_x - a_\theta b_r r_x\theta_y \\ &= a_r b_\theta(r_x\theta_y - r_y\theta_x) - a_\theta b_r(r_x\theta_y - r_y\theta_x) \\ &= (a_r b_\theta - a_\theta b_r)(r_x\theta_y - r_y\theta_x). \end{aligned} \tag{9}$$

Noting that

$$\begin{aligned} r_x\theta_y &= \tfrac{x}{\sqrt{x^2+y^2}} \tfrac{1}{1+\left(\frac{y}{x}\right)^2} \tfrac{1}{x} &= \tfrac{x^2}{r^3} \\ r_y\theta_x &= \tfrac{y}{\sqrt{x^2+y^2}} \tfrac{-yx^{-2}}{1+\left(\frac{y}{x}\right)^2} &= \tfrac{-y^2}{r^3}, \end{aligned}$$

and substituting into (9), the proof is complete.

□

Substituting (8) back into (7), we have

$$\begin{aligned} \phi(0) &= \frac{-1}{2\pi} \int_0^1 \int_0^{2\pi} \log|r|(a_r b_\theta - a_\theta b_r)\, d\theta dr \\ &= \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \log\left|\frac{1}{r}\right|(a_r b_\theta - a_\theta b_r)\, d\theta dr. \end{aligned}$$

Using the product rule,

$$\phi(0) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \log\left|\frac{1}{r}\right|((ab_\theta)_r - (ab_r)_\theta)\, d\theta dr.$$

Since $ab_r(0) = ab_r(2\pi)$,

$$\phi(0) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \log\left|\frac{1}{r}\right|(ab_\theta)_r\, d\theta dr,$$

then using integration by parts,

$$\phi(0) = \frac{1}{2\pi} \int_0^1 \frac{1}{r} \int_0^{2\pi} ab_\theta\, d\theta dr. \tag{10}$$

Now observe that

$$\int_0^{2\pi} ab_\theta d\theta = \int_0^{2\pi} (a - \overline{a}) b_\theta d\theta \quad \text{where } \overline{a}(r) = \frac{1}{2\pi} \int_0^{2\pi} a(r,\sigma) d\sigma.$$

Hence,

$$\begin{aligned} \left|\int_0^{2\pi} ab_\theta d\theta\right| &= \left|\int_0^{2\pi} (a - \overline{a}) b_\theta d\theta\right| \\ &\leq \|a - \overline{a}\|_{L^p(0,2\pi)} \|b_\theta\|_{L^q(0,2\pi)}\,, \end{aligned} \tag{11}$$

where the last inequality is true by Hölder's inequality.

By the Poincaré inequality, we have that

$$\|a - \overline{a}\|_{L^p(0,2\pi)} \|b_\theta\|_{L^q(0,2\pi)} \leq K_p \|a_\theta\|_{L^p(0,2\pi)} \|b_\theta\|_{L^q(0,2\pi)}\,, \tag{12}$$

$$\text{where} \quad K_p = \frac{p \sin(\pi/p)}{(p-1)^{1/p}}, \quad \text{see Appendix A.}$$

Now substituting (12) into (10), we have that

$$\begin{aligned}|\phi(0)| &\leq \frac{K_p}{2\pi}\int_0^1 \|a_\theta\|_{L^p}\|b_\theta\|_{L^q}\frac{dr}{r} \\ &= \frac{K_p}{2\pi}\int_0^1 \frac{\|a_\theta\|_{L^p}}{r^{\frac{1}{q}}}\frac{\|b_\theta\|_{L^q}}{r^{\frac{1}{p}}}dr.\end{aligned} \tag{13}$$

With another application of Hölder's inequality,

$$\begin{aligned}|\phi(0)| &\leq \frac{K_p}{2\pi}\left(\int_0^1 \left|\frac{\|a_\theta\|_{L^p}}{r^{\frac{1}{q}}}\right|^p dr\right)^{\frac{1}{p}}\left(\int_0^1 \left|\frac{\|b_\theta\|_{L^q}}{r^{\frac{1}{p}}}\right|^q dr\right)^{\frac{1}{q}} \\ &= \frac{K_p}{2\pi}\left(\int_0^1 \left|\frac{\int_0^{2\pi}|a_\theta|^p d\theta}{r^{\frac{p}{q}}}\right| dr\right)^{\frac{1}{p}}\left(\int_0^1 \left|\frac{\int_0^{2\pi}|b_\theta|^q d\theta}{r^{\frac{q}{p}}}\right| dr\right)^{\frac{1}{q}}.\end{aligned} \tag{14}$$

Note that $\frac{p}{q} = p-1$ and $\frac{q}{p} = q-1$, so we have

$$|\phi(0)| \leq \frac{K_p}{2\pi}\left(\int_0^1\int_0^{2\pi}\left|\frac{a_\theta}{r}\right|^p rd\theta dr\right)^{\frac{1}{p}}\left(\int_0^1\int_0^{2\pi}\left|\frac{b_\theta}{r}\right|^q rd\theta dr\right)^{\frac{1}{q}}. \tag{15}$$

Now observe that

$$\begin{aligned}\left(\int_0^1\int_0^{2\pi}\left|\frac{a_\theta}{r}\right|^p rd\theta dr\right)^{\frac{1}{p}} &= \left(\int_0^1\int_0^{2\pi}\left(\left|\frac{a_\theta}{r}\right|^2\right)^{\frac{p}{2}} rd\theta dr\right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1\int_0^{2\pi}\left(\left|\frac{a_\theta}{r}\right|^2 + |a_r|^2\right)^{\frac{p}{2}} rd\theta dr\right)^{\frac{1}{p}} \\ &= \left(\int_0^1\int_0^{2\pi}|\nabla a|^p rd\theta dr\right)^{\frac{1}{p}}.\end{aligned}$$

Thus:

$$\left(\int_0^1\int_0^{2\pi}\left|\frac{a_\theta}{r}\right|^p rd\theta dr\right)^{\frac{1}{p}} \leq \|\nabla a\|_{L^p(D^2)}. \tag{16}$$

Similarly,

$$\left(\int_0^1\int_0^{2\pi}\left|\frac{b_\theta}{r}\right|^q rd\theta dr\right)^{\frac{1}{q}} \leq \|\nabla b\|_{L^q(D^2)}. \tag{17}$$

Substituting (16) and (17) back into (15) we have

$$|\phi(0)| \leq \frac{K_p}{2\pi}\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)}. \tag{18}$$

This gives us the upper bound of ϕ at the center of the disk.

To find the upper bound for ϕ over the whole disk, we introduce the conformal transformation $T : D^2 \to D^2$ given by

$$T(z) = \frac{z_0 + z}{1 + \overline{z_0}z}, \qquad \text{with fixed } z_0 \in D^2. \tag{19}$$

T is a smooth map that maps the boundary of D^2 to itself, and $T(0) = z_0$.

Let

$$\tilde{a} = a \circ T, \quad \tilde{b} = b \circ T, \quad \widetilde{\phi} = \phi \circ T.$$

Lemma - Equation (5) is conformally invariant under T, namely:

$$\begin{cases} -\Delta\widetilde{\phi} & = \tilde{a}_x\tilde{b}_y - \tilde{a}_y\tilde{b}_x \\ \widetilde{\phi} & = 0. \end{cases} \tag{20}$$

Proof

Let $(u, v) = T(x, y)$.

$$\begin{aligned} -\Delta\widetilde{\phi}(x,y) &= -\Delta\phi(u,v) = -\phi(u,v)_{xx} - \phi(u,v)_{yy} \\ &= -(\phi_u u_x + \phi_v v_x)_x - (\phi_u u_y + \phi_v v_y)_y \\ &= -(\phi_{ux}u_x + \phi_u u_{xx}) - (\phi_{vx}v_x + \phi_v v_{xx}) - (\phi_{uy}u_y + \phi_u u_{yy}) - (\phi_{vy}v_y + \phi_v v_{yy}). \end{aligned}$$

Since T is a conformal transformation, then u and v satisfy the Cauchy-Riemann equations, (i.e. $u_x = v_y$ and $u_y = -v_x$), giving us,

$$-\Delta\widetilde{\phi} = -(u_x(\phi_{ux} + \phi_{vy}) + u_y(\phi_{uy} - \phi_{vx}) + \phi_u(u_{xx} + u_{yy}) + \phi_v(v_{xx} + v_{yy})),$$

Furthermore, since T is conformal, u and v are harmonic functions (ie. $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$), so

$$\begin{aligned} -\Delta\widetilde{\phi} &= -(u_x(\phi_{ux} + \phi_{vy}) + u_y(\phi_{uy} - \phi_{vx})) \\ &= -(u_x(\phi_{uu}u_x + \phi_{uv}v_x + \phi_{uv}u_y + \phi_{vv}v_y) + u_y(\phi_{uu}u_y + \phi_{uv}v_y - \phi_{uv}u_x - \phi_{vv}v_x)) \\ &= -(\phi_{uu}u_x^2 + \phi_{vv}u_x v_y + \phi_{uu}u_y^2 - \phi_{vv}u_y v_x) \\ &= -(\phi_{uu} + \phi_{vv})(u_x^2 + u_y^2) \\ &= -\Delta\phi(u_x^2 + u_y^2). \end{aligned} \tag{21}$$

Now evaluating the right hand side of (20),

$$\begin{aligned} \tilde{a}_x\tilde{b}_y - \tilde{a}_y\tilde{b}_x &= (a_u u_x + a_v v_x)(b_u u_y + b_v v_y) - (a_u u_y + a_v v_y)(b_u u_x + b_v v_x) \\ &= a_u u_x b_v v_y + a_v v_x b_u u_y - a_u u_y b_v v_x - a_v v_y b_u u_x \\ &= (a_u b_v - a_v b_u)(u_x v_y - u_y v_x) \\ &= (a_u b_v - a_v b_u)(u_x^2 + u_y^2). \end{aligned} \tag{22}$$

Combining (21) and (22) we see that

$$-\Delta\widetilde{\phi}(x,y) = \tilde{a}_x\tilde{b}_y - \tilde{a}_y\tilde{b}_x$$
$$\Rightarrow -\Delta\phi(u,v)(u_x^2+u_y^2) = (a_u b_v - a_v b_u)(u_x^2+u_y^2).$$

However $u_x^2+u_y^2 \geq 0$, and since $T' \neq 0$ then $u_x^2+u_y^2>0$, thus we have

$$-\Delta\phi(u,v) = a_u b_v - a_v b_u.$$

□

Using Green's Representation Theorem on (20) , similarly as in (7), we get

$$\widetilde{\phi}(0) = \frac{1}{2\pi}\int_0^1 \frac{1}{r}\int_0^{2\pi} \tilde{a}\tilde{b}_{\tilde{\theta}}d\tilde{\theta}dr. \tag{23}$$

T induces a diffeomorphism of ∂D^2. We paramaterize $\partial D^2 = T(\partial D^2)$ by $\tilde{\theta} \in [0,2\pi]$. There exists a constant $\gamma := \text{Arctan}(T(1))$ depending only on z_0 such that

$$\int_0^{2\pi} \tilde{a}\tilde{b}_{\tilde{\theta}}d\tilde{\theta} = \int_\gamma^{2\pi+\gamma} ab_\theta d\theta. \tag{24}$$

The Poincaré inequality is invariant under translation (mutatis mutandis Appendix A),

$$\left|\int_\gamma^{2\pi+\gamma} ab_\theta d\theta\right| \leq K_p\|a_\theta\|_{L^p(0,2\pi)}\|b_\theta\|_{L^q(0,2\pi)}\,, \tag{25}$$

hence

$$\left|\int_0^{2\pi} \tilde{a}\tilde{b}_{\tilde{\theta}}d\tilde{\theta}\right| \leq K_p\|a_\theta\|_{L^p(0,2\pi)}\|b_\theta\|_{L^q(0,2\pi)}\,.$$

Hence, as in (18),

$$|\widetilde{\phi}(0)| \leq \frac{K_p}{2\pi}\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)}\,.$$

Thus:

$$|\phi(z_0)| \leq \frac{K_p}{2\pi}\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)}\,.$$

As this holds for all z_0 in D^2 with the same constant K_p, we find

$$\|\phi\|_{L^\infty(D^2)} \leq \frac{K_p}{2\pi}\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)}\,. \tag{26}$$

Thus

$$C_\infty(p,D^2) = \frac{K_p}{2\pi},$$

thereby completing the proof.

□

Now we will prove the claim for the constant $C_2(p, D^2)$ in (4).

Theorem - Let ϕ be the solution to (5). Then we have:

$$\|\nabla\phi\|_{L^2(D^2)} \leq C_2(p, D^2)\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)},$$

with

$$C_2(p, D^2) = \sqrt{\frac{K_p}{2\pi}}, \quad \text{and } K_p = \frac{p\sin(\pi/p)}{(p-1)^{1/p}}.$$

Proof

Let

$$\nabla = (\partial_x, \partial_y), \quad \nabla^\perp = (-\partial_y, \partial_x).$$

Then

$$\nabla a \cdot \nabla b^\perp = a_x b_y - a_y b_x. \tag{27}$$

By Hölder's inequality,

$$\left\|\nabla a \cdot \nabla b^\perp\right\|_{L^1(D^2)} \leq \|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)}, \tag{28}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p < \infty.$$

Now we find an upper bound for $\|\nabla\phi\|_{L^2}$.

$$\begin{aligned}
\|\nabla\phi\|^2_{L^2(D^2)} &= \int_0^{2\pi}\int_0^1 |\nabla\phi|^2 r dr d\theta \\
&= \int_0^{2\pi}\int_0^1 (\phi_r^2 + \frac{1}{r^2}\phi_\theta^2) r dr d\theta \\
&= \int_0^{2\pi}\int_0^1 \phi_r^2 r dr d\theta + \int_0^{2\pi}\int_0^1 \frac{1}{r}\phi_\theta^2 dr d\theta.
\end{aligned} \tag{29}$$

Let

$$A = \int_0^{2\pi}\int_0^1 \phi_r^2 r dr d\theta \quad \text{and} \quad B = \int_0^{2\pi}\int_0^1 \frac{1}{r}\phi_\theta^2 dr d\theta.$$

First we simplify A. With an integration by parts, we get:

$$A = \int_0^{2\pi} \phi\phi_r r\Big|_{r=0}^{r=1} d\theta - \int_0^{2\pi}\int_0^1 \phi\phi_{rr} r dr d\theta - \int_0^{2\pi}\int_0^1 \phi\phi_r dr d\theta.$$

Then because $\phi = 0$ on ∂D^2,

$$A = -\int_0^{2\pi}\int_0^1 \phi\phi_{rr} r dr d\theta - \int_0^{2\pi}\int_0^1 \phi\phi_r dr d\theta.$$

Now we simplify B. Using integration by parts gives us

$$B = \int_0^1 \frac{1}{r}\phi\phi_\theta\Big|_{\theta=0}^{\theta=2\pi} dr - \int_0^1 \frac{1}{r}\int_0^{2\pi} \phi\phi_{\theta\theta} d\theta dr.$$

$\phi\phi_\theta(r,0) = \phi\phi_\theta(r,2\pi)$ so we have:

$$B = -\int_0^1 \frac{1}{r}\int_0^{2\pi} \phi\phi_{\theta\theta} d\theta dr.$$

Substituting A and B back into (29) we get

$$\begin{aligned} \|\nabla\phi\|^2_{L^2(D^2)} &= -\int_0^{2\pi}\int_0^1 \phi\phi_{rr}r + \frac{1}{r}\phi\phi_r + \frac{1}{r^2}\phi\phi_{\theta\theta} r dr d\theta \\ &= -\int_0^{2\pi}\int_0^1 \phi\Delta\phi r dr d\theta. \end{aligned} \tag{30}$$

By (27) ,

$$\|\nabla\phi\|^2_{L^2(D^2)} = \int_0^{2\pi}\int_0^1 \phi\nabla a\cdot\nabla b^\perp r dr d\theta,$$

and by Hölder's inequality,

$$\int_0^{2\pi}\int_0^1 \phi\nabla a\cdot\nabla b^\perp r dr d\theta \le \|\phi\|_{L^\infty(D^2)}\left\|\nabla a\cdot\nabla b^\perp\right\|_{L^1(D^2)}. \tag{31}$$

By (26) and (28),

$$\begin{aligned} \|\nabla\phi\|^2_{L^2(D^2)} &\le \|\phi\|_{L_\infty(D^2)}\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)} \\ &\le \frac{K_p}{2\pi}\|\nabla a\|^2_{L^p(D^2)}\|\nabla b\|^2_{L^q(D^2)}. \end{aligned}$$

Hence,

$$\|\nabla\phi\|_{L^2(D^2)} \le \sqrt{\frac{K_p}{2\pi}}\|\nabla a\|_{L^p(D^2)}\|\nabla b\|_{L^q(D^2)}. \tag{32}$$

Therefore we see that

$$C_2(p, D^2) = \sqrt{\frac{K_p}{2\pi}}, \quad \text{as claimed.}$$

□

3 Further work

Our results give the best constant for the disk but we do not know whether this is true for any Ω, unlike the standard Wente inequality [Topping (1997)]. Further, it is not clear whether the constant remains valid for Ω simply connected, unlike the standard Wente inequality [Baraket (1996)]. Inspecting our proof, we conjecture:

Let ϕ be the unique solution to (1), where $\partial\Omega$ is a graph over a circle, then it holds:

$$\begin{cases} \|\phi\|_{L^\infty(\Omega)} & \leq C_\infty(p,\Omega)\|\nabla a\|_{L^p(\Omega)}\|\nabla b\|_{L^q(\Omega)} \\ \|\phi\|_{L^2(\Omega)} & \leq C_2(p,\Omega)\|\nabla a\|_{L^p(\Omega)}\|\nabla b\|_{L^q(\Omega)}, \end{cases}$$

where $C_2(p,\Omega)$ and $C_\infty(p,\Omega)$ are as in (4).

This is because there exists a smooth, bijective, conformal map between all simply connected domains in $\mathbb{C}$, by the Riemann mapping theorem, taking the boundary of D^2 to the boundary of Ω. Since ∂D is mapped to $\partial\Omega$ in a one-to-one fashion and all points on $\partial\Omega$ can be parameterised along $[0,\ 2\pi]$ uniquely, this should not affect our constant found in (4).

Appendix A Poincaré Inequality

The Poincaré inequality, in one dimension on $(-1,1)$, is given by

$$\left\|a(x) - \bar{a}_x\right\|_{L^p(-1,1)} \leq G_p\left\|a'(x)\right\|_{L^p(-1,1)}, \tag{33}$$

where $\bar{a}_x = \frac{1}{2}\int_{-1}^{1} a(x)dx$.
Stanoyevitch (1990) found, on the interval $(-1,1)$, the Poincaré constant G_p to be

$$G_p = \frac{p\sin(\pi/p)}{\pi(p-1)^{1/p}}, \quad 1 < p < \infty.$$

We will show that on $(0,2\pi)$

$$\left\|a(y) - \bar{a}_y\right\|_{L^p(0,2\pi)} \leq \pi G_p\left\|a'(y)\right\|_{L^p(0,2\pi)}, \tag{34}$$

where $\bar{a}_y = \frac{1}{2\pi}a(y)dy$.

$$\tag{35}$$

Proof
Let

$$y = (x+1)\pi, \quad \frac{dy}{dx} = \pi.$$

We have

$$\begin{aligned} \left\|a(y) - \bar{a}_y\right\|^p_{L^p(0,2\pi)} &= \int_0^{2\pi}\left|a(y) - \frac{1}{2\pi}\int_0^{2\pi} a(u)du\right|^p dy \\ &= \pi\int_{-1}^{1}\left|a(\pi(x+1)) - \frac{1}{2}\int_{-1}^{1} a(\pi(u+1))du\right|^p dx. \end{aligned}$$

Let $A(x) = a(\pi(x+1))$ and $\bar{A}_x = \frac{1}{2}\int_{-1}^{1} A(u)du$, then

$$\begin{aligned}\left\|a(y) - \bar{a}_y\right\|_{L^p(0,2\pi)}^p &= \pi \int_{-1}^{1} \left|A(x) - \frac{1}{2}\int_{-1}^{1} A(u)du\right|^p dx \\ &= \pi\left\|A(x) - \bar{A}_x\right\|_{L^p(-1,1)}^p .\end{aligned}$$

By the Poincaré inequality,

$$\begin{aligned}\pi\left\|A(x) - \bar{A}_x\right\|_{L^p(-1,1)}^p &\leq \pi G_p^p \left\|A'(x)\right\|_{L^p(-1,1)}^p \\ &= \pi G_p^p \int_{-1}^{1} \left|a'(\pi(x+1))\right|^p dx.\end{aligned}$$

Now subsituting y in,

$$\begin{aligned}\pi G_p^p \int_{-1}^{1} \left|a'(\pi(x+1))\right|^p dx &= G_p^p \int_0^{2\pi} \left|\frac{da}{dy}\frac{dy}{dx}\right|^p dy \\ &= G_p^p \pi^p \left\|a'(y)\right\|_{L^p(0,2\pi)}^p \\ &= G_p^p \pi^p \int_0^{2\pi} |a'(y)|^p dy.\end{aligned}$$

Thus, as claimed,

$$\left\|a(y) - \bar{a}_y\right\|_{L^p(0,2\pi)} \leq G_p \pi \left\|a'(y)\right\|_{L^p(0,2\pi)} .$$

References

Baraket, S. (1996), 'Estimations of the best constant involving the L^∞ norm in Wente's inequality', *Annales de la faculté des sciences de Toulouse* **5**(3), 373–385.

Hélein, F. (2002), *'Harmonic Maps, Conservation Laws and Moving Frames'*, 2nd edn, Cambridge University Press, chapter 3, pp. 117–118.

Stanoyevitch, A. (1990), Geometry of Poincaré Domains, PhD thesis, The University of Michigan.

Topping, P. (1997), 'The optimal constant in Wente's L^∞ estimate', *Comment. Math. Helv.* **72**, 316–328.

Wente, H. (1969), 'An existence theorem for surfaces of constant mean curvature', *J. Math. Anal. Appl.* **26**, 318–344.